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Unified Equation of Motion of a Test Charge in Electromagnetic and Gravitational Fields

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This work starts by generalizing in a gravitational field the fundamental quantum mechanical commutation relations between the coordinates of a charged test particle and its momentum. Assuming that the components of the momentum of this test charge obey a noncommutative algebra in the presence of an electromagnetic field, it is proved that the commutator can be identified with the electromagnetic field tensor. Using these results, the equation of motion of this charged object in the presence of both the electromagnetic and gravitational fields is derived from their field equations. In this work, the laws of motion of a particle in the electromagnetic and gravitational fields has been unified with the field equations. Although the field equations themselves are not directly unified, this work strongly suggests that the scheme may act as a possible framework for the unification of at least gravitational and electromagnetic interactions.

1. INTRODUCTION

The Feynman–Dyson approach to electromagnetism (Dyson, 1990) has some remarkable similarities to Einstein's theory of gravitation. In the standard framework, Maxwell's field equations and the equation of motion of a charged particle are completely independent. However, due to the introduction of some commutation relations, Feynman obtained the most important result that the field equations of electromagnetism not only describe the spacetime behavior of the field, but also describe the motion of the charged objects in it. The laws of motion of the charges can be derived from the electromagnetic field equations and the Lorentz-covariant commutation relations in a $(3 + 1)$-dimensional flat spacetime. In this way, the theory unifies the field equations with the equations governing the motion of the charges in an electromagnetic field. A similar phenomenon

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occurs in Einstein's theory of gravitation (Weinberg, 1972). The equations of general relativity unify the gravitational field equations with the equation of motion of the sources creating this field. But due to the nonlinearity of Einstein's field equations, the test particle can move in its own gravitational field. However, Feynman and Dyson considered the electromagnetic field due to some external sources and that has been identified with the commutators between the components of the momentum of a test charge in it. It is implicitly assumed that the test charge does not alter the existing electromagnetic field and therefore the ultimate field equations remain the linear Maxwell equations.

In this paper we generalize the Feynman–Dyson theory to the presence of a gravitational field. The covariant commutators in the curved spacetime lead to unified equation of motion of test charges in electromagnetic and gravitational fields. This work may also be considered as a framework for a possible unified field theory. In Section 2 we find the relations between the fields and the commutators. In Section 3 we find the unified equation of motion of the test charge and in Section 4 we discuss our results.

In our notations $x^\mu = (ct, \mathbf{x})$ denotes the coordinates of the particle with respect to an observer in a $(3+1)$-dimensional pseudo-Riemannian, torsion-free spacetime with metric $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ (summation over the repeated indices is always implied over the range 0–3). The signature of the metric tensor $g_{\mu\nu}$ is $(+1, -1, -1, -1)$. c is the speed of light in vacuum. The proper time of the observer is calculated from $d\tau = (1/c)\, ds$.

2. FIELD EQUATIONS OF GRAVITATION AND ELECTROMAGNETISM

Suppose a particle of rest mass m_0 and charge q exists at x^μ with respect to an observer in some pseudo-Riemannian spacetime. p_μ is its momentum. Following Dyson (1990), we assume the following commutation relations are obeyed by the particle:

$$[x^\mu, x^\nu] = 0 \tag{2.1}$$

$$[x^\mu, p_\nu] = -i\hbar\delta^\mu{}_\nu \tag{2.2}$$

As the spacetime coordinates follow a commutative algebra, the general theory of relativity holds good in this spacetime. The equations for the gravitational field which curves the spacetime will be given by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.3}$$

where the energy-momentum-stress tensor $T_{\mu\nu}$ is due to some external sources not including the test particle, by definition. The definitions of symmetric affine connection and Ricci curvature tensor are the usual ones (Weinberg, 1972):

$$\Gamma^{\mu}{}_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}) \tag{2.4}$$

$$R_{\mu\nu} = \partial_\mu \partial_\nu(\ln\sqrt{-g}) - \partial_\alpha \Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\mu\nu}\partial_\alpha(\ln\sqrt{-g}) \tag{2.5}$$

where $g = \det(g_{\alpha\beta})$ and $\partial_\alpha \equiv \partial/\partial x^\alpha$. As

$$[p_\alpha, g^{\mu\nu}] \neq 0 \tag{2.6}$$

a suitable ordering convention is needed for defining the contravariant components of the momentum, $p^\mu = m_0\, dx^\mu/d\tau$. Using the standard (normal) symmetric ordering convention, one can define

$$p^\mu = \tfrac{1}{2}(p_\alpha g^{\alpha\mu} + g^{\alpha\mu} p_\alpha) \tag{2.7}$$

Applying (2.1) and (2.2), we find

$$[x^\mu, p^\nu] = -i\hbar g^{\mu\nu} \tag{2.8}$$

It can be shown (Hojman and Shepley, 1991) that $[x^\mu, p^\nu]$ is symmetric in μ, ν indices and therefore the above result is consistent.

Hence the solution of Einstein's equations with proper boundary conditions will uniquely give the metric tensor of the spacetime, which can be identified with the commutators between the components of coordinates x^μ and momentum p^ν of the test charge.

Define an antisymmetric field $F_{\mu\nu}$ as

$$F_{\mu\nu}(x, p) = \frac{ic}{\hbar q}[p_\mu, p_\nu] \tag{2.9}$$

We shall show that $F_{\mu\nu}$ satisfies the sourceless Maxwell equations in this curved spacetime. The definition suggests that in general $F_{\mu\nu}$ depends on both x^μ and p_ν. However, using the Jacobi identity

$$[x^\alpha, [p_\mu, p_\nu]] + [p_\mu, [p_\nu, x^\alpha]] + [p_\nu, [x^\alpha, p_\mu]] = 0 \tag{2.10}$$

with equations (2.2) and (2.9), one obtains

$$[x^\alpha, F_{\mu\nu}(x, p)] = 0 \tag{2.11}$$

If $F_{\mu\nu}(x, p)$ is analytic in at least some finite domain $\mathbb{D}(x, p)$ of x^μ and p_ν, then from (2.1) and (2.2) (Merzbacher, 1970)

$$[x^\alpha, F_{\mu\nu}(x, p)] = -i\hbar \frac{\partial F_{\mu\nu}}{\partial p_\alpha} \tag{2.12}$$

in $\mathbb{D}(x, p)$. So equation (2.11) implies that $F_{\mu\nu}$ can be chosen independent of p_μ in $\mathbb{D}$, that is, $F_{\mu\nu} = F_{\mu\nu}(x)$. Another application of the Jacobi identity gives

$$[p_\alpha, [p_\beta, p_\gamma]] + [p_\beta, [p_\gamma, p_\alpha]] + [p_\gamma, [p_\alpha, p_\beta]] = 0 \tag{2.13}$$

which by virtue of (2.9) gives

$$[p_\alpha, F_{\beta\gamma}] + [p_\beta, F_{\gamma\alpha}] + [p_\gamma, F_{\alpha\beta}] = 0 \tag{2.14}$$

Since $F_{\mu\nu}(x)$ is analytic in $\mathbb{D}$, equation (2.2) implies (Merzbacher, 1970)

$$[p_\alpha, F_{\beta\gamma}(x)] = i\hbar\, \partial_\alpha F_{\beta\gamma} \tag{2.15}$$

Therefore (2.14) reduces to

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{2.16}$$

in the domain $\mathbb{D}$. These are the sourceless Maxwell equations in flat spacetime. In curved spacetime the covariant derivative of $F_{\mu\nu}$ is defined by

$$D_\mu F_{\alpha\beta} = \partial_\mu F_{\alpha\beta} - \Gamma^\nu{}_{\mu\alpha} F_{\nu\beta} - \Gamma^\nu{}_{\mu\beta} F_{\alpha\nu} \tag{2.17}$$

From the symmetry of the affine connection $\Gamma^\alpha{}_{\mu\nu}$ in μ, ν and the antisymmetry of $F_{\alpha\beta}$ it can be shown that (Weinberg, 1972)

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} \tag{2.18}$$

Hence (2.16) becomes

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0 \tag{2.19}$$

These are the well-known sourceless Maxwell equations in the presence of a gravitational field (Weinberg, 1972). Therefore we see that the commutators between the components of the momentum of the test charge in a gravitational field identically satisfy the sourceless Maxwell equations. If $J^\mu(x)$ is the source for the electromagnetic field, then the other Maxwell equations

$$J^\mu(x) = \frac{c}{4\pi} D_\mu F^{\mu\nu} \tag{2.20}$$

will determine the commutators uniquely.

Hence the solution of Maxwell's equations in the curved spacetime with proper boundary conditions will give the electromagnetic field tensor which can be identified with the commutators between the components of the momentum p_μ of the test charge.

3. EQUATION OF MOTION OF THE TEST PARTICLE

The test charge q of rest mass m_0 is moving in the electromagnetic and gravitational fields. All the commutators between the components of coordinates and momentum of the test charge are now known. Therefore, using them and the field equations, we can derive the differential equation of motion of this test charge.

Let us define the force on the particle due to the fields as

$$K_\mu(x, p) = \frac{dp_\mu}{d\tau} \tag{3.1}$$

From the definitions (2.7) of p^μ and (2.9) of $F_{\mu\nu}$, we find

$$[p^\mu, p_\nu] = -\frac{i\hbar q}{c} F^\mu{}_\nu(x) - \frac{i\hbar}{2} (p_\alpha \partial_\nu g^{\alpha\mu} + \partial_\nu g^{\alpha\mu} p_\alpha) \tag{3.2}$$

where the domain in which the metric tensor $g_{\mu\nu}(x)$ is analytic has been considered and the following result has been used (Merzbacher, 1970):

$$[p_\alpha, g^{\mu\nu}] = i\hbar\, \partial_\alpha g^{\mu\nu} \tag{3.3}$$

Now differentiating equation (2.2) with respect to the proper time τ and then applying (3.2) we obtain

$$\frac{im_0}{\hbar} [x^\mu, K_\nu] + \frac{q}{c} F^\mu{}_\nu = -\frac{1}{2} (p_\alpha \partial_\nu g^{\alpha\mu} + \partial_\nu g^{\alpha\mu} p_x) \tag{3.4}$$

This is the equation of motion of the charged particle in commutator form. Applying the standard (normal) symmetric ordering convention, we can define the total derivative of $F_{\mu\nu}(x)$ as

$$\frac{dF_{\mu\nu}(x)}{d\tau} = \frac{1}{2m_0} (p_\alpha \partial^\alpha F_{\mu\nu} + \partial^\alpha F_{\mu\nu} p_\alpha) \tag{3.5}$$

Applying this definition to the differential equations (2.9), we also find the following consistency conditions:

$$[p_\mu, K_\nu] - [p_\nu, K_\mu] = -\frac{i\hbar q}{2m_0 c} (p_\alpha \partial^\alpha F_{\mu\nu} + \partial^\alpha F_{\mu\nu} p_\alpha) \tag{3.6}$$

Hence to obtain the differential equation of motion of the test charge one has to find the most general solution of (3.4) subject to the consistency conditions (3.6). The most general solution of $K_\nu(x, p)$ will be of the form

$$K_\nu(x, p) = -\frac{q}{m_0 c}(ap_\alpha F^\alpha{}_\nu + bF^\alpha{}_\nu p_\alpha) - \frac{1}{m_0}(Pp_\alpha p_\beta \partial_\nu g^{\alpha\beta} + Qp_\alpha \partial_\nu g^{\alpha\beta} p_\beta + R\,\partial_\nu g^{\alpha\beta} p_\alpha p_\beta) + H_\nu(x, p) \tag{3.7}$$

where a, b, P, Q and R are in general complex numbers independent of x^s and p_ν. They satisfy the following equations:

$$a + b = 1 \tag{3.8a}$$

$$2P + Q = \tfrac{1}{2} \tag{3.8b}$$

$$Q + 2R = \tfrac{1}{2} \tag{3.8c}$$

$H_\mu(x, p)$ are arbitrary functions of x^μ and p_ν which satisfy the following commutation relations:

$$[x^\mu, H_\nu(x, p)] = 0 \tag{3.9}$$

To determine a, b, P, Q and R uniquely, the solution of $K_\nu(x, p)$ is substituted in the consistency conditions (3.6). Using the definition (2.9) of $F_{\mu\nu}$ and equations (2.15) and (3.3), we obtain the following result from the general solution of K_ν:

$$\begin{aligned}
&[p_\mu, K_\nu] - [p_\nu, K_\mu] \\
&\quad = -\frac{i\hbar qa}{m_0 c} p_\alpha(g^{\beta\alpha}\partial_\mu F_{\beta\nu} + g^{\beta\alpha}\partial_\nu F_{\mu\beta} + F_{\beta\nu}\partial_\mu g^{\alpha\beta} - F_{\beta\mu}\partial_\nu g^{\alpha\beta}) \\
&\quad\quad - \frac{i\hbar qb}{m_0 c}(\partial_\mu F_{\beta\nu} g^{\beta\alpha} + \partial_\nu F_{\mu\beta} g^{\beta\alpha} + F_{\beta\nu}\partial_\mu g^{\alpha\beta} - F_{\beta\mu}\partial_\nu g^{\alpha\beta})p_\alpha \\
&\quad\quad + \frac{i\hbar qP}{m_0 c}(F_{\mu\alpha} p_\beta \partial_\nu g^{\alpha\beta} - F_{\nu\alpha} p_\beta \partial_\mu g^{\alpha\beta} + p_\alpha F_{\mu\beta}\partial_\nu g^{\alpha\beta} - p_\alpha F_{\nu\beta}\partial_\mu g^{\alpha\beta}) \\
&\quad\quad + \frac{i\hbar qQ}{m_0 c}(F_{\mu\alpha}\partial_\nu g^{\alpha\beta} p_\beta - F_{\nu\alpha}\partial_\mu g^{\alpha\beta} p_\beta + p_\alpha \partial_\nu g^{\alpha\beta} F_{\mu\beta} - p_\alpha \partial_\mu g^{\alpha\beta} F_{\nu\beta}) \\
&\quad\quad + \frac{i\hbar qR}{m_0 c}(\partial_\nu g^{\alpha\beta} F_{\mu\alpha} p_\beta - \partial_\mu g^{\alpha\beta} F_{\nu\alpha} p_\beta + \partial_\nu g^{\alpha\beta} p_\alpha F_{\mu\beta} - \partial_\mu g^{\alpha\beta} p_\alpha F_{\nu\beta}) \\
&\quad\quad + [p_\mu, H_\nu] - [p_\nu, H_\mu]
\end{aligned} \tag{3.10}$$

Applying the consistency conditions (3.6) and the electromagnetic field equations (2.16), we find

$$\frac{im_0 c}{\hbar q}([p_\mu, H_\nu] - [p_\nu, H_\mu])$$
$$= \left(\frac{1}{2} - a\right) p_\alpha \partial^\alpha F_{\mu\nu} + \left(\frac{1}{2} - b\right) \partial^\alpha F_{\mu\nu} p_\alpha$$
$$+ (a - P - Q)\, p_\alpha (F_{\nu\beta} \partial_\mu g^{\alpha\beta} - F_{\mu\beta} \partial_\nu g^{\alpha\beta})$$
$$+ (b - Q - R)(F_{\nu\beta} \partial_\mu g^{\alpha\beta} - F_{\mu\beta} \partial_\nu g^{\alpha\beta})\, p_\alpha$$
$$+ P(F_{\mu\alpha} p_\beta \partial_\nu g^{\alpha\beta} - F_{\nu\alpha} p_\beta \partial_\mu g^{\alpha\beta}) + R(\partial_\nu g^{\alpha\beta} p_\beta F_{\mu\alpha} - \partial_\mu g^{\alpha\beta} p_\alpha F_{\nu\alpha}) \tag{3.11}$$

$H_\mu(x, p)$ are arbitrary functions that arise due to the integration of equation (3.4) and are not a consequence of the existing electromagnetic and gravitational fields. To maintain this arbitrariness we must have

$$[p_\mu, H_\nu(x, p)] - [p_\nu, H_\mu(x, p)] = 0 \tag{3.12}$$

because otherwise H_μ will have to depend on the fields and their derivatives. Equation (3.12) will be valid if all the coefficients of the r.h.s. of (3.11) identically vanish. Then

$$a = \tfrac{1}{2} = b \tag{3.13a}$$

$$a - P - Q = 0 = b - Q - R \tag{3.13b}$$

$$P = 0 = R \tag{3.13c}$$

These equations uniquely determine the numbers a, b, P, Q and R as

$$a = b = Q = \tfrac{1}{2} \tag{3.14a}$$

$$P = R = 0 \tag{3.14b}$$

which are consistent with equations (3.8a)–(3.8c).

Therefore the differential equation of motion of the test charge in presence of both electromagnetic and gravitational fields has been found to be

$$m_0 \frac{dp_\mu}{d\tau} + \frac{1}{2} p_\alpha \partial_\mu g^{\alpha\beta} p_\beta = -\frac{q}{2c}(p_\alpha F^\alpha{}_\mu + F^\alpha{}_\mu p_\alpha) + H_\mu(x, p) \tag{3.15}$$

where H_μ are arbitrary functions which satisfy the commutation relations (3.9) and (3.12). When there are no other fields, $H_\mu(x, p)$ can be chosen to be zero. Then we get the well-known Einstein–Maxwell equation of motion.

4. REMARKS AND DISCUSSION

It appears from Dyson (1990) that Feynman's original motivation was to find more general methods than the usual quantization program, to find new ways of looking at physics. Our motivation for generalizing his techniques in curved spacetime was to see how the gravitational field can be brought into the electromagnetic interactions and in this way to achieve a somewhat unified theory. Although in this paper the electromagnetic and gravitational fields are not entirely unified, our theory leads to a unified equation of motion for a massive, charged test particle in the presence of both of those fields. Also the field equations for electromagnetism and gravitation have been linked through these laws of motion.

The work of Feynman and Dyson has also been generalized by Lee (1990) for non-Abelian electrodynamics. It has been found that Feynman's scheme fits very well in the Yang–Mills theory. Although his original aims were different, considering all these results, it appears that Feynman and Dyson's theory may stand as a possible framework for establishing a unified theory of electricity, magnetism, and gravity. In this connection it should also be noted that it is not possible to obtain an expression for $H_\mu(x, p)$, which appeared in the equation of motion of the test charge, only from the information of the electromagnetic and gravitational fields. H_μ was chosen arbitrarily to be zero, but it will be extremely interesting if H_μ contains information of the weak and strong fields interacting with the test particle.

It is important to note that the bracket $[A, B]$ in our calculations may not be a strictly quantum mechanical commutator. In general, any arbitrary Lie bracket can work that satisfies equations (2.1) and (2.2) identically.

For defining p^μ and $dF_{\mu\nu}/d\tau$ we used the standard (normal) symmetric ordering convention, which is not necessarily important. By defining

$$p^\mu = \xi p_\alpha g^{\alpha\mu} + \eta g^{\alpha\mu} p_\alpha \tag{4.1}$$

$$\frac{dF_{\mu\nu}}{d\tau} = \frac{1}{m_0} (\chi p_\alpha \partial^\alpha F_{\mu\nu} + \lambda \partial^\alpha F_{\mu\nu} p_\alpha) \tag{4.2}$$

where ξ, η, χ, λ are all in general complex numbers, and applying the following equations due to their corresponding classical formulas

$$\xi + \eta = 1 \tag{4.3a}$$

$$\chi + \lambda = 1 \tag{4.3b}$$

one can get the same results as the symmetric (normal) ordering convention. The consistency condition on the force $K_\mu(x, p)$ uniquely determines the values of all the numbers as 1/2, which implies the symmetric ordering convention.

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